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SOME OPEN PROBLEMS IN THE THEORY OF LOCALLY CONVEX SPACES

A. Aytuna *P. B. Djakov* *A. P. Goncharov* *T. Terzioğlu*
V. P. Zahariuta

It is well known that a mathematical theory is alive (i.e. it is an active research area) if it has some fundamental open problems. In that sense the theory of locally convex spaces (lcs) is is still not dead - it has challenging open problems and our aim here is to present some of them together with a collection of related questions, hypotheses and open problems put by the participants of the seminar. We hope that this paper will stimulate more mathematicians to do research in the theory of locally convex spaces.

0. PRELIMINARIES

Since most of the problems we consider are connected with (nuclear) Fréchet spaces (NFS) let us remind that typical examples of NFS are spaces of the kind $C^\infty(D)$ (C^∞ - functions on an open domain $D \subset \mathbb{R}^n$), $C^\infty(\bar{D})$ (C^∞ - functions on an open bounded domain D, which are uniformly continuous together with all derivatives), $\mathcal{E}(K)$ (Whitney functions on a compact $K \subset \mathbb{R}^n$), $A(G)$ (holomorphic functions on an open domain $G \subset \mathbb{C}^n$) .

A sequence (e_n) in a lcs E is called **basis** if any element $x \in E$ has an unique representation as a convergent series of the kind

$$x = \sum_{n=1}^{\infty} x_n e_n,$$

where x_n are scalars. A basis (e_n) is called **absolute** if for each continuous seminorm $\|.\|$ on E the condition

$$\sum_{n=1}^{\infty} |x_n| \|e_n\| < \infty$$

holds. If E is a Fréchet space with an absolute basis (e_n) and a fundamental system of seminorms $\|.\|_p$, $p = 1, 2, \ldots$, then it is isomorphic to the Köthe space

$$K(a_{np}) = \{\xi = (\xi_n)_{n=1}^{\infty} : \|\xi\|_p = \sum_{n=1}^{\infty} a_{np}|\xi_n| < \infty \ \forall p\},$$

where $a_{np} = \|e_n\|_p$. In partiqular, due to the Dynin - Mityagin theorem on absoluteness of bases in NFS, any nuclear Fréchet space with a basis is isomorphic to a Köthe space in a canonical fashion.

For example the space $C^{\infty}[-1,1]$ is isomorphic to the Köthe space $s = K(n^p)$ (see [30]), the space $A(\mathbb{D})$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, is isomorphic to $K(\exp(-n/p))$, the space $A(\mathbb{C})$ is isomorphic to $K(\exp(pn))$. In both spaces $A(\mathbb{D})$ and $A(\mathbb{C})$ the monomials $z^n, n = 0, 1, 2, \ldots$, form a basis and the corresponding representation of a function $f \in A(\mathbb{D})$ or $f \in A(\mathbb{C})$ as a series with respect to the basis coincides with the Taylor series of f. As a generalization of these examples Köthe spaces of the kind

$$E_0(a) = K(\exp(-\frac{1}{p}a_n)) \ , \ E_{\infty}(a) = K(\exp(pa_n)),$$

where $a = (a_n)$ is a sequence of positive numbers, are called power series spaces of finite or infinite type. The space s is obviously infinite type power series space since $s = K(\exp(p \log n))$.

Suppose X is a Fréchet space and $\|.\|_p, p = 1, 2, \ldots$ be a system of seminorms generating the topology of X. The following interpolation properties (which are obviously invariant under isomorphisms) define very important classes of Fréchet spaces (Vogt [47], Vogt and Wagner [52], Zahariuta [64], [65]):

$$\exists p \ \forall q \ \exists r \ \exists C \ | \ \|x\|_q^2 \leq C\|x\|_p\|x\|_r, \quad x \in X; \qquad (1)$$

$$\forall p \ \exists q \ \forall r \ \exists C \ | \ (\|x'\|^*)_q^2 \leq C\|x'\|_p^*\|x'\|_r^*, \quad x' \in X'; \qquad (2)$$

$$\forall p \ \exists q \ \forall r \ \exists \varepsilon \ \exists C \ | \ (\|x'\|^*)_q \leq C(\|x'\|_p^*)^{\varepsilon}(\|x'\|_r^*)^{1-\varepsilon}, \quad x' \in X'; \qquad (3)$$

$$\exists p \ \forall q \ \exists r \ \exists \varepsilon \ \exists C \ | \ \|x\|_q \leq C\|x\|_p^{\varepsilon}\|x\|_r^{1-\varepsilon}, \quad x' \in X'. \qquad (4)$$

This notions were preceeded by some versions for Köthe spaces (Dragilev [15], Bessaga [5], Zahariuta [60], Terzioğlu [41], Dubinsky [17]). Let us note that for these classes D.Vogt uses respectively the notations DN, $\bar{\Omega}$, Ω, $\underline{DN}$, while V. Zahariuta uses the notations D_1, D_2, Ω_1, Ω_2.

1. GENERAL PROBLEMS ON LCS

1.1. Existence of Bases in NFS

The problem of Grothendieck of existence of bases in NFS was solved negatively by Mityagin and Zobin [69]. However still there are some problems concerning existence of bases.

Problem 1 *Is it true that any complemented subspace in a NFS with a basis has a basis?*

This problem was posed by Pelczynski and in general it is still open. Mityagin [31] proved that the answer is positive for finite power series spaces, but for infinite power series spaces the answer is not known. Important partial results have been obtained by Dubinsky and Vogt [18], [19] (yes - for tame infinite power series spaces), Wagner [53] (yes - if F is a complemented subspace of s such that $F \times F \simeq F$,) Aytuna, Krone and Terzioğlu [3] (yes, if F is a complemented subspace of s, such that F and $F \times F$ have equal diametral dimensions). In connection with the last paper we consider the following

Problem 2 *The existence of a basis in F (where F is a complemented subspace of s having the same diametral dimension as its cartesian square) was proved in [3] by using the decomposition method of Pelczynski in the version due to Vogt. Is it possible to give a constructive proof of this result?*

It is known for a long time (see, for example, Mityagin [30], Zahariuta [66], Mityagin and Henkin [32]) that so called "Dead-end space method" gives us a possibility to build a basis . Recently it became clear that the existence of a "Dead-end space" is also a necessary condition (see [13], [26], [27]) where for a given NFS E with a fundamental system of norms $\|\cdot\|_p, , p = 1, 2, \ldots$ some criterions for existence of a basis are obtained in terms of existence of Hilbert spaces $(H_0, \|\cdot\|_0)$ and $(H_\infty, \|\cdot\|_\infty)$ (dead-end space) with special relations between norms $\|\cdot\|_0, \|\cdot\|_\infty$ and $\|\cdot\|_p, , p = 1, 2, \ldots$ and also between the corresponding unit balls (or dual norms). However these criterions for existence of a basis are not inner. Therefore it is natural to state the following

Problem 3 *Does there exist an inner criterion (i.e. which involves only the given system of norms) for existence of a bases in NFS?*

Let us note that papers of Vogt [48], [49] may be useful in attacking Problem 2. In [48] Vogt prove the following inner characterization of finite power series spaces : *The space $E, \|\cdot\|_p$ is isomorphic to a finite power series space if and only if $E \in (\underline{DN}) \cap \bar{\Omega}$*

We expect that if there is an inner criterion for existence of a basis then it should also possess two conditions (analogous to $(\underline{DN})$ and $(\bar{\Omega})$ in terms of some interpolation properties of the given fundamental system of norms and the corresponding dual norms. Of course any such criterion will give a solution to the Pelczynski FP 1. Let us remark that since $\bar{\Omega}$ is inherited by subspaces while $(\underline{DN})$ by quotient spaces the Vogt's characterization of finite power series spaces gives immediately another proof of the Mityagin's result that any complemented subspace of a finite power series space has a basis.

Let us mention also another interesting characterization due to Vogt [48]: A Fréchet space X is isomorphic to a complemented subspace of the space (s) if and only if $X \in (DN) \cap (\Omega)$. In this connection of course the following question arises:

Problem 4 *Does the condition $X \in (DN) \cap (\Omega)$ imply that the space X has a basis?*

In order to state our next question suppose E is a NFS with a basis $\{e_i, i \in I\}$ and $I = \cup I_n$, where the subsets $I_n, n = 1, 2, \ldots$ are disjoint. Then we obtain block decomposition $E = \oplus_n E_n$ of E, where the n-th block E_n is the closed linear hull of the vectors $\{e_i, i \in I_n\}$. The subspace $H \subset E$ is called **block subspace** if $H = \oplus_n H \cap E_n$. It is known [12] that if $\sup \dim E_n < \infty$ then any complemented block subspace is isomorphic to some subspace, generated by a part of the given basis, i.e. it not only has a basis but also satisfies Bessaga conjecture (see the next subsection).

Problem 5 *Is it true that any complemented block subspace has a basis if* $\dim E_n < \infty$, *but* $\sup \dim E_n = \infty$ *?*

Since having a basis in the space E is equivalent to the existence of one-dimensional projections $P_n, n = 1, 2, \ldots,$ such that

$$x = \sum_{n=1}^{\infty} P_n x \qquad \forall x \in E \qquad \text{and } P_n P_m = 0 \text{ for } n \neq m,$$

it is natural to consider the question for existence of a sequence P_n of finite-dimensional projections satisfying the same conditions. Further such a sequence of projections is called finite-dimensional decomposition. It is known (see Djakov and Mityagin [11]) that there are NFS without finite-dimensional decomposition satisfying $\sup \dim P_n < \infty$. Moreover there exist examples of NFS without bounded approximation property, so without finite dimensional decomposition ([34], [50]). However still there is no answer to the following question (cf. [11]):

Problem 6 *Does there exist a NFS E such that for any continuous projection $P : E \longrightarrow E$ we have either* $\dim P < \infty$ *or* $\dim(I - P) < \infty$ *?*

At the end of this section let us note that all known constructions of NFS without basis (see [69], [11], [34], [50]) are rather artificial. Still there is no example of a functional NFS without basis. In this connection we state the following

Problem 7 *Does there exists a functional NFS without basis?*

In the section devoted to concrete function spaces we give some comments to this problem.

1.2. Quasiequivalence of Bases

If E is a NFS with a basis (e_i) then one can obtain a new basis in E by the following three operations: **rearrangement** of the basis with a permutation $\sigma : N \to N$, **multiplication** with scalars $\rho_i \neq 0$ and transformation of the basis with an isomorphism $T : E \to E$. Therefore it is possible to consider the question of uniqueness of basis in NFS only up to application of a rearrangement, a multiplication with scalars and an isomorphism. In accordance to this observation two bases (e_i) and (f_i) in the space E are called **quasiequivalent** if there exist a permutation $\sigma : N \to N$, scalars $\rho_i \neq 0$ and an isomorphism $T : E \to E$ such that $f_i = T(\rho_i e_{\sigma(i)})$, $i \in N$.

The first important result on quasiequivalence of bases in NFS was obtained by Dragilev [14], who proved that all bases in the space of analytic functions on the unit disc $A(D)$ are quasiequivalent. Generalizing his proof Mityagin [30] showed the quasiequivalence of bases in nuclear power series spaces (nuclear Hilbert scales). The further progress is connected with the works of Dragilev [15], Bessaga [5], Zaharuita [56],[61], Mityagin [31], Crone and Robinson [10], Kondakov [25] and many other mathematicians. As a result for wide classes of spaces was proved that all bases are quasiequivalent (for more information see the book of Dragilev [16]). Let us only mention here (see [10],[25]) that all bases in E are quasiequivalent if the space E has a regular basis (i.e. E has a basis (e_i) such that for some fundamental system of seminorms $||.||_p$, $p = 1, 2, \ldots$, the ratio $||e_i||_p/||e_i||_{p+1}$ is decreasing with respect to i for any fixed p.) However still is open

Problem 8 *Is it true that any two bases in a NFS are quasiequivalent?*

The following more general form of the quasiequivalence problem, is known as Bessaga conjecture [5] :

Problem 9 *Is it true that if E is a NFS with a basis (e_i) and F is a complemented subspace of E with a basis (f_j), then (f_j) is quasiequivalent to a part of the basis (e_i)?*

This problem has been solved positively in some special cases (see, for example [5], [12], [26], [35]). Let us note that obviously the Bessaga conjecture is connected also with Problem 1. Namely, the positive answer to Problem 1 for a space E having the quasiequivalence property implies obviously that Bessaga conjecture holds for this space.

In order to consider some related questions for tensor products let us remind that spaces of the form

$$E(\lambda, c) := K(\exp(-1/p + \lambda_i p)c_i),$$

where $c = (c_i), c_i > 0, \lambda = (\lambda_i), 0 < \lambda_i \leq 1$, are called **power spaces of the first kind** ([61]) It is easy to check that spaces of the kind

$$E_0(a) \times E_\infty(b), \quad E_0(a) \hat{\otimes} E_\infty(b)$$

are isomorphic to power spaces of the first kind.

Let $X = K(a_{ip}, i \in I)$ and $Y = K(b_{jp}, j \in J)$ be Köthe spaces. An operator $T : X \to Y$ is called **quasidiagonal** if there exist a function $\varphi : I \to J$ and constants r_i, $i \in I$ such that

$$Te_i = r_i \tilde{e}_{\varphi(i)}, \quad i \in I,$$

where (e_i) and $(\tilde{e}_j)$ are the canonical bases in the spaces X and Y. We denote respectively by $X \overset{qd}{\simeq} Y$ a quasidiagonal isomorphism between the spaces X and Y.

Conjecture. Let L be a complemented subspace of a space $E_0 \hat{\otimes} E_\infty$, then L has a basis and, moreover, L is isomorphic to some power space of the first kind, i.e. $L \simeq E(\lambda, c)$ for some λ and μ. More generally, any complemented subspace in a power space of the first kind is isomorphic to a power space of the first kind.

Problem 10 *To prove the above conjecture about the structure of complemented subspaces if it is known a priory that a subspace L has a basis.*

Let us notice that any basis subspace (step subspace) L of the space $E_0 \hat{\otimes} E_\infty$, (i.e. $L = span\{e_i \otimes e_j, (i,j) \in \nu \subset \mathbb{N}^2\}$) is isomorphic to a power space of the first kind.

Problem 11 *To prove the conjecture under some additional restrictions on a "size" of L (for example $L \simeq L^2$) but without a priori assumption about existence of a basis in L.*

Conjecture. Let L be a complemented subspace of a space of the kind

$$X = E_\infty(a) \hat{\otimes} E'_\infty(b),$$

then L has a basis and moreover, L is isomorphic to a mixed F, DF space of the following special form:

$$G(\lambda, c) = \lim_{p\to\infty} proj \lim_{q\to\infty} ind\, l^1(\exp(-q\lambda_i + p)c_i),$$

where $c = (c_i), c_i > 0, \lambda = (\lambda_i), 0 < \lambda_i \leq 1$.

Valdivia [46] showed that the above conjecture is true under some strong constraints (for example, if $X = s\hat{\otimes}s'$ and L contains additionally complemented subspace M, which is isomorphic to $s\hat{\otimes}s'$). In such a way he get as a partial case a solution of the well known Grothendieck's problem (see [23]) on existence of a basis in the space $\mathcal{O}_{\mathcal{M}}$.

Analogical questions can be considered for tensor products of another pairs of power series spaces and duals of power series spaces.

Conjecture. If L is a complemented subspace of a space of the kind

$$(a) \quad E_0(a)\hat{\otimes}E'_\infty(b); \quad (b) \quad E_\infty(a)\hat{\otimes}E'_0(b); \quad (c) \quad E_0(a)\hat{\otimes}E'_0(b),$$

then L is isomorphic to a mixed F, DF-space of the corresponding type:

$$(a) \quad H(\lambda, c) = \lim_{p\to\infty} proj \lim_{q\to\infty} ind\, l^1(\exp(-q\lambda_i - \frac{1}{p})c_i);$$

$$(b) \quad K(\lambda, c) = \lim_{p\to\infty} proj \lim_{q\to\infty} ind\, l^1(\exp(p\lambda_i + \frac{1}{q})c_i);$$

$$(c) \quad L(\lambda, c) = \lim_{p\to\infty} proj \lim_{q\to\infty} ind\, l^1(\exp(\frac{1}{q} - \frac{1}{p}\lambda_i)c_i).$$

1.3. Linear Topological Invariants

One of important questions is whether two given linear topological spaces are isomorphic or not. To answer such a question it is useful to deal with some properties of linear topological spaces which are invariant under isomorphisms. More precisely, if Σ is a class of linear topological spaces, Ω is a set with a relation of equivalence $\sim$ and $\Phi : \Sigma \to \Omega$ is a mapping, such that

$$X \simeq Y \Rightarrow \Phi(X) \sim \Phi(Y),$$

then Φ is called LTI. We say that the invariant Φ is complete on the class Σ if for any $X, Y \in \Sigma$

$$\Phi(X) \sim \Phi(Y) \Rightarrow X \simeq Y.$$

First LTI connected with isomorphic classification of Fréchet spaces are due to A.N.Kolmogorov [24] and A.Pelczynski [39]. They introduced LTI called **approximative**

dimensions and proved by their help that $A(D) \not\simeq A(G)$ if $D \subset \mathbb{C}^n$, $G \subset \mathbb{C}^m$, $n \neq m$, and $A(\mathbb{D}^n) \not\simeq A(\mathbb{C}^n)$, where $\mathbb{D}^n$ is the unit polydisc in $\mathbb{C}^n$. Later C.Bessaga, A.Pelczynsky and S.Rolewicz [4] and B. Mityagin [30] considered other LTI called **diametral dimensions,** which turns out to be stronger and more convenient than the approximative dimensions. For example, the diametral dimension

$$\Gamma'(X) = \{\gamma = (\gamma_n) : \exists p \, \forall q \quad \frac{\gamma_n}{d_n(U_q, U_p)} \to 0\},$$

is a complete invariant in the class of all Fréchet spaces with absolute regular basis and proving this fact (independently) Crone and Robinson [10] and Kondakov [25] show that nuclear Fréchet spaces with basis have the quasiequivalence property. However it is easy to see that outside the class of spaces with regular basis the diametral dimensions are not complete invariants. For example, in general diametral dimensions do not distinguish spaces of the kinds

$$E_0(a) \times E_\infty(b), \quad E_0(a) \hat{\otimes} E_\infty(b).$$

Using some ideas of Mityagin [31] Zahariuta (see [59], [61], [62]) introduced a new series of invariants, which are convenient for investigation of isomorphic classification of classes of Köthe spaces without regular basis. In particular he considers for a power series space $X = E(\lambda, a)$ the characteristic functions

$$\begin{aligned} \alpha_1(\delta;\, \tau, t) &= |\{i : \delta < \lambda_i,\ \tau < a_i \le t\}| \\ \alpha_2(\delta;\, \tau, t) &= |\{i : \delta > \lambda_i,\ \tau < a_i \le t\}| \\ \alpha_3(\delta, \varepsilon;\, \tau, t) &= |\{i : \delta < \lambda_i \le \varepsilon,\ \tau < a_i \le t\}| \end{aligned}$$

It turns out that these functions are invariant characteristics of the space X in the following sense:

If the spaces $X = E(\lambda, a)$ and $\tilde{X} = E(\tilde{\lambda}, \tilde{a})$ are isomorphic then

$$\begin{aligned} &(i) \quad \forall \delta' \ \exists \delta, c \ : \ \alpha_1(\delta;\, \tau, t) \le \tilde{\alpha_1}(\delta';\, \tau/c, ct) \\ &(ii) \quad \forall \delta \ \exists \delta', c \ : \ \alpha_2(\delta;\, \tau, t) \le \tilde{\alpha_2}(\delta';\, \tau/c, ct) \\ &(iii) \quad \forall \varepsilon' \ \exists \varepsilon \quad \forall \delta \ \exists \delta', c \ : \ \alpha_3(\delta, \varepsilon;\, \tau, t) \le \tilde{\alpha_3}(\delta', \varepsilon';\, \tau/c, ct) \end{aligned}$$

and also the symmetric relations hold (obtained by interchanging the roles of $\lambda, \tilde{\lambda}$ and $a, \tilde{a}$).

Since the above characteristics are built by considering the set of indices i for which λ_i and a_i belong to some intervals (so the pairs (λ_i, a_i) belong to some rectangle) we call the corresponding LTI **one-rectangle invariant**. Zahariuta [61] proved that if $X \overset{qd}{\simeq} X^2$ and $\tilde{X} \overset{qd}{\simeq} \tilde{X}^2$ then the one-rectangle invariant is complete, i.e. the above conditions imply $X \simeq \tilde{X}$. Moreover, the corresponding isomorphism is quasidiagonal and in such a way was proved the quasiequivalence property for any power space of the first kind which is isomorphic to its square.

Yurdakul and Zahariuta [55] proved that the one-rectangle invariant is complete on the class of all shift-stable spaces of the kind $E_0(c) \times E_\infty(d)$. Djakov and Zahariuta showed that without the restriction of shift-stability the one-rectangle invariant may not distinguish nonisomorphic spaces of the above-mentioned kind (unpublished).

Problem 12 *To describe the widest class of power spaces of the first kind on which the one-rectangle invariant is complete.*

It is worth to note that to obtain a criterion of quasidiagonal isomorphism for spaces $E(\lambda, a)$, $E(\tilde{\lambda}, \tilde{a})$ we need to consider uniform estimates (not depending on m) of m-rectangle characteristics. Namely the following fact is true:

Proposition. $E(\lambda, a) \overset{qd}{\simeq} E(\tilde{\lambda}, \tilde{a})$ *if and only if there exist* $c > 0$ *and* $\varphi : (0,1] \to (0,1]$, $\varphi(1) = 1$, $\varphi(t) \downarrow 0$ *if* $t \downarrow 0$, *such that*

$$|\bigcup_{k=1}^{m} \{i : \delta_k < \lambda_i \le \varepsilon_k;\ \tau_k < a_i \le t_k\}| \le$$

$$|\bigcup_{k=1}^{m} \{i : \varphi(\delta_k) < \tilde{\lambda}_i \le \varphi^{-1}(\varepsilon_k);\ \frac{\tau_k}{c} < \tilde{a}_i \le c t_k\}|, \tag{5}$$

$$|\bigcup_{k=1}^{m} \{i : \delta_k < \tilde{\lambda}_i \le \varepsilon_k;\ \tau_k < \tilde{a}_i \le t_k\}| \le$$

$$|\bigcup_{k=1}^{m} \{i : \varphi(\delta_k) < \lambda_i \le \varphi^{-1}(\varepsilon_k);\ \frac{\tau_k}{c} < a_i \le c t_k\}|, \tag{6}$$

for all collections of parameters $0 \le \delta_k \le \varepsilon_k < 1$, $1 \le \tau_k \le t_k < \infty$, $k = 1, \dots, m$ *and for arbitrary* $m \in \mathbb{N}$.

Problem 13 *Is the statement of the above Proposition true if we replace the quasidiagonal isomorphism* $\overset{qd}{\simeq}$ *by the usual isomorphism* $\simeq$?

A positive answer gives us the quasiequivalent property for power spaces of the first kind. But it is not known even the answer of the following weaker question.

Problem 14 *Let* $E(\lambda, a) \simeq E(\tilde{\lambda}, \tilde{a})$. *Do the estimates* (5), (6) *hold with* c *and* φ *depending on* m, *if* $m \ge 2$?

Analogous problems arise for the mixed F, DF-spaces $G(\lambda, c)$, $H(\lambda, c)$, $K(\lambda, c)$, $L(\lambda, c)$, considered above.

Let us note that Chalov [8] and Chalov and Zahariuta [9] investigate joint isomorphisms of triples of weighted l^2-spaces and obtained results which may be useful in solving the above problem. They constructed for any $m = 1, 2, \dots$ an **m-rectangle** invariant on the class of triples of weighted l^2-spaces which is strictly stronger then the corresponding $(m-1)$-rectangle invariant, $m > 2$, in the sense there exists an example of two triples which are distinguished by the m-rectangle invariant but are not distinguished by the $(m-1)$-rectangle invariant. We guess an analogous result for the power spaces of the first kind.

1.4. Normability Conditions on Fréchet Spaces

Let E be a Fréchet space throughout with a basis of neighborhoods of zero $U_1 \supset U_2 \supset U_3 \supset \dots$. E is **quasinormable** if for each n there is a k such that the topologies induced on U_n^o by $E_b^{'}$ and by the Banach space $E^{'}[U_k^o]$ coincide. E has the **density condition** if the bounded subsets of $E_b^{'}$ are metrizable. E is **distinguished** if $E_b^{'}$ can be expressed as the inductive limit of a sequence of Banach spaces. Thus a quasinormable Fréchet space has the density condition and if E has the density condition then it is also distinguished. The definitions given here are not always the usual definitions. For example, Meise and Vogt [33] have shown that E is quasinormable if and only if it satisfies an Ω-type condition. Hence quasinormability is inherited by quotient spaces but not subspaces. Motivated by the results of Bonet and Diaz [7] in [36] the following result was proved.

Theorem. *Every subspace of E has the density condition if and only if E is a Montel space or E is isomorphic to a subspace of $X \times \omega$, where X is some Banach space.*

An immediate corollary is that if every subspace of E is quasinormable, then either E is a Schwartz space or E is isomorphic to a subspace of $X \times \omega$, where X is again a Banach space. These results and some results of Bonet and Diaz [7] suggest the following question.

Problem 15 . *If every quotient space of E has the density condition, is E quasinormable?*

We can formulate the problem treated in the theorem for other related classes of Fréchet spaces.

Problem 16 . *Characterize E if every subspace of E is distinguished.*

If E satisfies a DN-type condition, then it is **asymptotically normable** [42]. E satisfies the normability condition (y) if

$$E' = \bigcup_{k=1}^{\infty} \overline{E'[U_1^\circ] \cap U_k^\circ},$$

where the closure is taken with respect to a topology of pairing (E, E') (cf. [36] and its references). An asymptotically normable Fréchet space satisfies (y). Also, if E has the bounded approximation property and admits a continuous norm then it satisfies (y). There is an example of a Köthe space which is not even locally normable [42]. Hence bounded approximation property does not imply asymptotical normability. However the following question seems to be yet unanswered.

Problem 17 . *Let E be a Schwartz space which satisfies (y). Does E have the bounded approximation property?*

One should note that a Fréchet Schwartz space which satisfies (y) is asymptotically normable and also quasinormable. If we restrict our attention to a narrower class of spaces, we have the following question.

Problem 18 . *Let E be a nuclear space with (DN) and (Ω). Does E have the finite dimensional decomposition property?*

Of course one should mention that this problem is related to Problem 4.

2. PROBLEMS ON CONCRETE FUNCTION SPACES

2.1. Spaces of Analytic Functions

Let G be an open domain in $\mathbb{C}^n$ (or in some complex manifold) and $A(G)$ denote the space of analytic functions on G with its natural topology of uniform convergence on compact subsets; then it is a nuclear Fréchet space. If $E \subset \mathbb{C}^n$ is not open, then $A(E)$ denotes the space of analytic germs on E with the inductive limit topology, i.e.

$$A(E) = \lim_{G} ind\, A(G),$$

where G varies in the class of all open domains containing E.

Problem 19 *Does there exists an open domain G such that the space $A(G)$ has no basis?*

Of course this problem is a special case of Problem 7. For domains $G \subset \bar{\mathbb{C}}$ it is known that the space $A(G)$ has a basis in the following three cases:

1. (a) G is a regular domain in $\mathbb{C}$ (Walsh [54], Leja [29]) ;

2. (b) ∂G is a polar set in $\bar{\mathbb{C}}$ (Zahariuta [57]) ;

3. (c) $G = G_1 \cap G_2$, $\partial G_1 \cup \partial G_2 = \emptyset$ and G_1 satisfies (a), G_2 satisfies (b).

For a long time as a possible example of a space without basis was considered a space of the kind $A(D)$, where

$$D = \bar{\mathbb{C}} \setminus (\bigcup_{p=1}^{\infty} K_s \ \cup \{a\}), \quad K_s = \{z \in \bar{\mathbb{C}} : |z - a_s| \le \delta_s\}, \quad a = \lim_s a_s,$$

and a is an irregular point for the domain G in the sense of Potential Theory. It is still not known what is the answer to the following

Problem 20 *Does there exists a basis in the space $A(D)$?*

As it was shown (see [57]) the space $A(D)$ is not isomorphic to any of the spaces

$$A_0 = A(\{z \in \mathbb{C} : |z| < 1\}),\ A_\infty = A(\mathbb{C}),\ A_0 \times A_\infty \simeq A(\{z \in \mathbb{C} : 0 < |z| < 1\})$$

if the point a is not regular.

It is interesting to consider the more general question on isomorphic classification of spaces of the kind $A(D)$ by using some appropriate LTI.

Conjecture. *There exists a continuum of pairwise nonisomorphic spaces among the spaces $A(D)$.*

Suppose $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc in the complex plane , and $F \neq \emptyset$ is a proper subset of $\partial \mathbb{D}$.

Problem 21 *Does the space $A(\mathbb{D} \cup F)$ have a basis?*

Let us note that the answer of this problem is not known even in the case F consists of only one point.

Problem 22 *Suppose F_1 and F_2 are subsets of $\partial \mathbb{D}$. Characterize the isomorphism $A(\mathbb{D} \cup F_1) \simeq A(\mathbb{D} \cup F_2)$ in terms of the sets F_1 and F_2.*

It is known that $A(\mathbb{D} \cup F_1) \not\simeq A(\mathbb{D} \cup F_2)$ in the following two cases (joint result of V. Erofeev and V. Zahariuta, unpublished): 1) F_1 is one point set, F_2 is an arc; 2) one of the sets F_1, F_2 is closed while the other is open in the topology of $\partial \mathbb{D}$.

Let us note that in the multidimensional case an analogous problem (with an arbitrary complete n-circular domain D insead of $\mathbb{D}$ and n-circular subsets F_1, F_2 of ∂D) turns to be considerably more simple since the system of monomials z^α, $\alpha \in \mathbb{Z}_+^n$, is a natural basis in the corresponding spaces of analytic functions.

In the framework of Analitic Function Spaces Problem 2 has a positive solution. In fact a special case of the result given in [3] shows that for a Stein manifold M if the space $A(M)$ has the property DN (see (1)) then it has a basis. Moreover in the case when a Stein manifold M admits a continuous plurisubharmonic exhaustion function which is maximal (i.e. satisfying the homogeneous Complex Monge-Ampere equation) off a compact set a direct proof of existence of basis in the space $A(M)$ of analytic functions on M can be given (see [1], [2]). This class of Stein manifolds includes open Riemann surfaces such that $A(M)$ has the property (DN), i.e. Parabolic Riemann surfaces. In this context the following natural question arises ([2], [66], 2.2.3).

Problem 23 *Is it true that every Stein manifold M, $dim M > 1$, with $A(M)$ having the property (DN) possesses a plurisubharmonic exhaustion function which is maximal off a compact set.*

At the end let us note that many other problems connected with spaces of analytic functions can be found in [66].

2.2. Spaces of C^∞-Functions

Let D be an open bounded set in $\mathbb{R}^n$ and $C^\infty(\bar{D})$ denote the space of all C^∞-functions which derivatives are uniformly continuous in D. Considered with the system of norms

$$|f|_p = \sup\{|\partial^\alpha f(x)|,\ x \in D,\ |\alpha| \le p\}$$

$C^\infty(\bar{D})$ is a Fréchet space.

It is known that if the boundary of the domain D is smooth [45], Lipshitz [67], or even Hölder [43], [51], [21], then the space $C^\infty(\bar{D})$ is isomorphic to the space s of rapidly decreasing sequences. On the other hand it is known that there exists a continuum of pairwise nonisomorphic spaces (Tidten [44], Goncharov and Zahariuta [20]) of the type $C^\infty(\bar{D}_\psi)$, where

$$D_\psi = \{(x,y) \in \mathbb{R}^2 \ :\ x \in (0,1),\ |y| < \psi(x)\},$$

is a domain with a cusp, which sharpness depends on how fast $\psi(x)$ tends to 0 as $x \to 0$.

For a long time spaces of the kind $C^\infty(\bar{D}_\psi)$ have been considered as a possible example of a space of C^∞-functions without basis. However bases were built recently in some subspaces $C_F^\infty(\bar{D}_\psi)$, consisting of functions, which are flat in the point $(0,0)$, under some assumptions on the function ψ [28],[22]. These subspaces have similar linear topological spaces (for example, there exists continuum of pairwise nonisomorphic such spaces).

But it is still open the following

Problem 24 *Is it true that any space $C^\infty(\bar{D}_\psi)$ has a basis?*

The usual method for construction of bases in the spaces of the kind $C^\infty(\bar{D})$ in the case D is a domain with enough smooth boundary consists in the following: take an appropriate Hilbert space $H \supset C^\infty(\bar{D})$, orthogonalize monomials in H and prove that the expansion of any function $f \in C^\infty(\bar{D})$ with respect to the obtained orthogonal system is converging in $C^\infty(\bar{D})$. On the other hand bases in the spaces of the kind $C_s^\infty(\bar{D}_\psi)$ were constructed by using another rather complicated way.

Problem 25 *Is it possible to find an natural Hilbert space (for example, some weighted Sobolev space) such that a basis in $C_F^\infty(\bar{D}_\psi)$ can be obtained by orthogonalization of monomials in H ?*

But it seems that answer would be negative even if we consider the following more general question.

Problem 26 *Could $C_F^\infty(\bar{D}_\psi)$ have a polynomial basis?*

Let $\mathcal{E}(K)$ be the space of Whitney functions on a compact set $K \subset \mathbb{R}^n$ and let Δ be a cube containing K. We regard $\mathbb{R}^n$ as a subset of $\mathbb{C}^n$ and denote by $g_K(z)$ the corresponding generalized Green function with the pole in ∞ (see [66]). Let us consider the following conditions:

(1) $\exists M, C : \sup_K |\partial^\alpha P(x)| \leq C.(deg P)^{M|\alpha|} \sup_K |P|$

for any polynomial P $\forall \alpha$ (Markov inequality for polynomials);

(2) there exists linear continuous extension operator $S : \mathcal{E}(K) \to \mathcal{E}(\Delta)$;

(3) the spaces $\mathcal{E}(K)$ and (s) are isomorphic.

It is known (see [43], [51], [21], [38], [40], [68]) that these conditions are equivalent if K is a perfect compact set. For their realization it is sufficient the next property of the function g_K : $K \neq \emptyset$

$$(4) \quad \exists C, \delta > 0 \quad : \quad g_K(z) \leq C(dist(K,z))^\delta \quad \forall z \in \mathbb{C}^n.$$

Problem 27 *Do the conditions (1) - (3) imply (4). If not, is it possible to give a criterion for existence of an extension operator or a criterion for (1)-(3) in terms of g_K or an other characteristic function of K ?*

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